Resit Exam — Functional Analysis (WBMA033-05)

Wednesday 26 June 2024, 8.30h–10.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

The linear space $\mathcal{C}([0,1],\mathbb{K})$ can be equipped with the following norms:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| \quad \text{and} \quad \|f\|_* = \sup_{x\in[0,1]} \frac{1}{1+x^2}|f(x)|.$$

Are these norms equivalent? Motivate your answer.

Problem 2 (5 + 5 + (5 + 10) = 25 points)

Consider the following Banach space over $\mathbb{C}$:

$$X = \left\{ f : \mathbb{R} \to \mathbb{C} \;:\; \sup_{x\in\mathbb{R}} |f(x)| < \infty \right\}, \quad \|f\| = \sup_{x\in\mathbb{R}} |f(x)|.$$

Consider the following linear operator:

$$T : X \to X, \quad Tf(x) = \frac{f(x+1) + f(x-1)}{2}$$

(a) Show that T is bounded.

(b) Show that the function $f_a(x) = e^{iax}$, with $a \in \mathbb{R}$, is an eigenvector of T and that the corresponding eigenvalue is a real number.

(c) Show in two different ways that T is *not* compact:

  (i) By using properties of $\sigma(T)$.

  (ii) By considering the sequence (Tf_n) for a suitably chosen sequence (f_n) in X.

Problem 3 (10 + 5 = 15 points)

Let X be a Hilbert space over $\mathbb{C}$ and assume that $u \in X$ satisfies $\|u\| = 1$.

(a) Show that $Px = \langle x, u\rangle u$ is an orthogonal projection.

(b) Show that the operator $T = I - 2P$ satisfies $\|Tx\| = \|x\|$ for all $x \in X$.

Turn page for problems 4 and 5!

Problem 4 (5 + 10 = 15 points)

(a) Formulate the Closed Graph Theorem.

(b) Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Prove that the following statements are equivalent:

  (i) T is bounded;

  (ii) for any sequence (x_n) in X such that $x_n \to 0$ and $Tx_n \to y$ we have $y = 0$.

Problem 5 (5 + 10 + 10 = 25 points)

(a) Formulate the Hahn-Banach Theorem for normed linear spaces.

(b) Equip the space $\mathcal{C}([0,1],\mathbb{K})$ with the sup-norm and consider the following linear maps:

$$f : \mathcal{C}([0,1],\mathbb{K}) \to \mathbb{K}, \qquad f(\varphi) = \int_0^1 \varphi(t)\,dt,$$

$$g : \mathcal{C}([0,1],\mathbb{K}) \to \mathbb{K}, \qquad g(\varphi) = \varphi(\tfrac{1}{2}).$$

Show that $\|f\| = 1$ and $\|g\| = 1$.

(c) For the functions $\varphi(t) = 1 - t$ and $\psi(t) = t$ let $V = \operatorname{span}\{\varphi, \psi\}$ and consider the linear map

$$h : V \to \mathbb{K}, \qquad h(a\varphi + b\psi) = \frac{a+b}{2}.$$

What can we say about uniqueness when we apply the Hahn-Banach Theorem to h?

End of test (90 points)

Solution of problem 1 (10 points)

The the two norms are equivalent if there exist constants $0 < m \leq M$ such that

$$m\|f\|_\infty \leq \|f\|_* \leq M\|f\|_\infty \quad \text{for all} \quad f \in \mathbb{C}([0,1],\mathbb{K}).$$

(4 points)

Take an arbitrary $f \in \mathbb{C}([0,1],\mathbb{K})$. For all $x \in [0,1]$ we have

$$\frac{1}{2}|f(x)| \leq \frac{1}{1+x^2}|f(x)| \leq |f(x)|.$$

(3 points)

Taking the supremum over all $x \in [0,1]$ gives

$$\frac{1}{2}\|f\|_\infty \leq \|f\|_* \leq \|f\|_\infty.$$

Since $f \in \mathbb{C}([0,1],\mathbb{K})$ is arbitrary, it follows that the norms are equivalent and the constants are given by $m = 1/2$ and $M = 1$.
(3 points)

Solution of problem 2 (5 + 5 + (5 + 10) = 25 points)

(a) For any $x \in [0, 1]$ we have

$$\begin{aligned} |Tf(x)| &\leq \frac{1}{2}(|f(x+1)| + |f(x-1)|) \\ &\leq \frac{1}{2}(\|f\|_\infty + \|f\|_\infty) \\ &\leq \|f\|_\infty. \end{aligned}$$

(3 points)

Taking the supremum over all $x \in [0, 1]$ gives

$$\|Tf\|_\infty = \sup_{x \in [0,1]} |Tf(x)| \leq \|f\|_\infty,$$

which shows that the operator T is bounded.
(2 points)

(b) We have that

$$\begin{aligned} Tf_a(x) &= \frac{f_a(x-1) + f_a(x+1)}{2} \\ &= \frac{e^{ia(x-1)} + e^{ia(x+1)}}{2} \\ &= \frac{e^{ia} + e^{-ia}}{2} e^{iax} \\ &= \cos(a) f_a(x), \end{aligned}$$

which shows that f_a is an eigenvector of T.
(4 points)

The associated eigenvalue $\lambda = \cos(a)$ is indeed a real number.
(1 point)

(c) (i) Compact operators can have at most countably many eigenvalues. But from part (b) it follows that T has uncountably many eigenvalues (namely at least all values $\lambda \in [-1, 1]$). Therefore, T cannot be compact.
(5 points)

(ii) For each $n \in \mathbb{N}$ define the following function:

$$f_n(x) = \begin{cases} 1 & \text{if } x = n, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence (f_n) belongs to X and is bounded as $\|f_n\| = 1$ for all $n \in \mathbb{N}$.
(4 points)

But if $n \neq m$ we have $\|Tf_n - Tf_m\| = \|f_{n+1} - f_{m+1}\| = 1$.
(3 points)

Therefore, the sequence (Tf_n) does not have a convergent subsequence. We conclude that T cannot be compact.
(3 points)

Solution of problem 3 (10 + 5 = 15 points)

(a) For any $x \in X$ we have

$$\begin{aligned} P^2x &= \langle Px, u\rangle u \\ &= \langle \langle x, u\rangle u, u\rangle u \\ &= \langle x, u\rangle \|u\|^2 u \\ &= \langle x, u\rangle u \\ &= Px, \end{aligned}$$

which implies $P^2 = P$ and thus that P is a projection.
(6 points)

To show that P is an *orthogonal* projection, we can proceed in two different ways.

Method 1. Note that $(\operatorname{ran} P)^\perp = (\operatorname{span}\{u\})^\perp = \ker P$, which implies by definition that P is an orthogonal projection.
(4 points)

Method 2. A projection on a Hilbert space is orthogonal if and only if it is selfadjoint. For all $x, y \in X$ we have

$$\begin{aligned} \langle Px, y\rangle &= \langle \langle x, u\rangle u, y\rangle \\ &= \langle x, u\rangle \langle u, y\rangle \\ &= \langle x, u\rangle \overline{\langle y, u\rangle} \\ &= \langle x, \langle y, u\rangle u\rangle \\ &= \langle x, Py\rangle, \end{aligned}$$

which shows that $P = P^*$.
(4 points)

(b) *Method 1.* Since P is an orthogonal projection we have

$$\operatorname{ran}(I - P) = \ker P = (\operatorname{ran} P)^\perp.$$

Let $x \in X$ be arbitrary. Applying the Pythagorean theorem twice gives

$$\begin{aligned} \|Tx\|^2 &= \|(I - P)x - Px\|^2 \\ &= \|(I - P)x\|^2 + \|Px\|^2 \\ &= \|(I - P)x + Px\|^2 = \|x\|^2. \end{aligned}$$

(5 points)

Method 2. For any $x \in X$ we have

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle x - 2Px, x - 2Px\rangle \\ &= \langle x - 2\langle x, u\rangle u, x - 2\langle x, u\rangle u\rangle \\ &= \langle x, x\rangle - 2\langle x, u\rangle\langle u, x\rangle - 2\langle u, x\rangle\langle x, u\rangle + 4|\langle x, u\rangle|^2\langle u, u\rangle \\ &= \langle x, x\rangle - 4|\langle x, u\rangle|^2 + 4|\langle x, u\rangle|^2 = \|x\|^2. \end{aligned}$$

(5 points)

Solution of problem 4 (5 + 10 = 15 points)

(a) Let X and Y be Banach spaces, let $V \subset X$ be a closed linear subspace, and let $T : V \to Y$ be a linear map. If the graph of T is closed, then $T \in B(V, Y)$.
(5 points)

(b) Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Prove that the following statements are equivalent:

(i) T is bounded;

(ii) for any sequence (x_n) in X such that $x_n \to 0$ and $Tx_n \to y$ we have $y = 0$.

Proof of (i) ⇒ (ii). Assume that T is bounded. Let (x_n) be a sequence such that $x_n \to 0$ and $Tx_n \to y$. Then it follows that

$$\|y\| = \|y - Tx_n + Tx_n\| \leq \|y - Tx_n\| + \|Tx_n\| \leq \|y - Tx_n\| + \|T\| \, \|x_n\|.$$

Since the right-hand side tends to zero, it follows that $y = 0$.
(5 points)

Proof of (ii) ⇒ (i). Assume that $x_n \to x$ and $Tx_n \to y$. Introduce the new sequence $z_n = x_n - x$. Then it follows that $z_n \to 0$ and $Tz_n \to y - Tx$. By assumption it follows that $y - Tx = 0$ so that $y = Tx$.
(3 points)

We conclude that the graph of T is closed. Since X and Y are Banach spaces we can apply the Closed Graph Theorem with $V = X$ to conclude that T is bounded.
(2 points)

Solution of problem 5 (5 + 10 + 10 = 25 points)

(a) Let X be a normed linear space and let $V \subset X$ be a linear subspace. If $f \in V'$, then there exists $F \in X'$ such that $F(v) = f(v)$ for all $v \in V$ and $\|F\| = \|f\|$.
(5 points)

(b) For $\varphi \in \mathcal{C}([0,1], \mathbb{K})$ we have that

$$|f(\varphi)| = \left| \int_0^1 \varphi(t)\, dt \right| \leq \int_0^1 |\varphi(t)|\, dt \leq \int_0^1 \|\varphi\|_\infty\, dt = \|\varphi\|_\infty.$$

(4 points)

For $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|f(\varphi)| = 1$. Hence,

$$\|f\| = \sup_{\varphi \neq 0} \frac{|f(\varphi)|}{\|\varphi\|_\infty} = 1.$$

(1 point)

For $\varphi \in \mathcal{C}([0,1], \mathbb{K})$ we have that

$$|g(\varphi)| = |\varphi(\tfrac{1}{2})| \leq \sup_{x \in [0,1]} |\varphi(x)| = \|\varphi\|_\infty.$$

(4 points)

For $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|g(\varphi)| = 1$. Hence,

$$\|g\| = \sup_{\varphi \neq 0} \frac{|g(\varphi)|}{\|\varphi\|_\infty} = 1.$$

(1 point)

(c) First observe that $f(\rho) = g(\rho) = h(\rho)$ for all $\rho \in V$.
(2 points)

In particular, it then follows that

$$\|h\| = \sup_{\rho \in V \setminus \{0\}} \frac{|h(\rho)|}{\|\rho\|_\infty} = \sup_{\rho \in V \setminus \{0\}} \frac{|f(\rho)|}{\|\rho\|_\infty} \leq \sup_{\rho \in \mathcal{C}([0,1],\mathbb{K}) \setminus \{0\}} \frac{|f(\rho)|}{\|\rho\|_\infty} = \|f\| = 1.$$

For $\rho(t) = 1$ we have $\|\rho\|_\infty = 1$ and $|h(\rho)| = 1$, which implies that $\|h\| = 1$.
(4 points)

We conclude that both f and g are norm preserving extensions of h. But note that $f \neq g$, since for $\rho(t) = t^2$ we have $f(\rho) = \frac{1}{3}$ whereas $g(\rho) = \frac{1}{4}$. Therefore, the norm preserving extension of h obtained by the Hahn-Banach Theorem is not unique.
(4 points)